A CONVEX SOLUTION TO RECEDING HORIZON CONTROL OF SWITCHED LINEAR SYSTEMS

BY

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THESIS

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Abstract

The uniform stabilization of discrete-time switched linear systems subject to a strongly connected switching constraint is exactly solved for finite-path-dependent controllers with finite horizon knowledge of future switching modes. Conditions for the existence of both a full-information state-feedback controller and a dynamic output feedback controller are given in the form of finite-dimensional systems of linear matrix inequalities. Controller synthesis is accomplished with no unnecessary assumptions by solving any feasible system of linear matrix inequalities from an increasing sequence of such families.

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Chapter 1

Introduction

This thesis considers the uniform stabilization of discrete-time switched linear systems. Switched linear systems are those whose parameters vary within a finite set, where each element in the set represents a mode of operation. The mode of the system is governed by a switching signal subject to a strongly connected switching constraint. Switched systems are often used as abstractions of hybrid automata in which system dynamics are affected by discrete events [15, 12]. In particular, when a switched system has complex switching dynamics, they are often replaced by a nondeterministic switching signal For this reason, analyzing the stability of such systems and designing stabilizing controllers have been studied often [4, 5].

When the switching signal is unconstrained (i.e. at each time the system may take on any mode in the parameter set) it is alternatively called a discrete linear inclusion. Checking the stability of such a system is equivalent to showing that the joint spectral radius of the parameter set is less than one [11, 4]. This problem is only semidecidable - when the system is indeed stable this fact can be verified in finite time using piecewise Lyapunov function methods [5]. This approach has previously been used to characterize uniform stabilization and disturbance attenuation for switched linear system using controllers which are finite-path-dependent [14, 13].

Stabilization of switched linear systems has also been considered using a finite horizon control scheme [1, 8, 16]. These results typically involve solving a finite-horizon optimization problem at each time step to determine the control effort. However, these methods provide no guarantee of performance as the underlying optimization problems may have no solution.

This thesis provides exact conditions for the uniform stabilization of switched linear systems via finite-path-dependent controllers with access to a finite receding horizon of future switching modes. This characterization comes in the form of an increasing sequence of families of linear matrix inequalities. This controller synthesis reduces to the design problem solved in [14] when the future horizon length is taken to be zero, and the conditions specified here are identical to these past results in this case. Solving any feasible family of inequalities within this sequence allows for the synthesis of a switched controller which depends on a finite interval of the switching signal.

The developments in this thesis are organized as follows. In Chapter 2 mathematical results and a summary of the stability of linear time-varying systems are presented to support the work that follows. Chapter 3 examines the stability of autonomous switched systems and presents a characterization via a finite family of Lyapunov inequalities. The autonomous result is then applied to the closed-loop stability of controlled systems to develop linear conditions for the existence of a stabilizing controller. Existence conditions are presented for both full-information state feedback controllers and then for outback feedback. Examples are given in Chapter 4 to demonstrate the controller synthesis technique as well as show the benefit of a controller using a finite horizon over controllers without such a horizon. In addition the use of software to systematically search for stabilizing controllers is briefly discussed. Concluding remarks and a projection of future work are presented in Chapter 5.

Chapter 2

Background Material

This chapter presents a summary of the mathematical concepts and system theory used in the developments that follow. Results that are used directly are presented here, along with their proofs. References which provide more detailed background are provided as needed.

The following notation will be used throughout: for any matrix $X \in \mathbb{R}^{n\times m}$ the image, kernel, and rank of X are represented by $\operatorname{Im} X$, $\operatorname{Ker} X$, and $\operatorname{rank} X$ respectively. The notation $N(X)$ refers to any full-rank matrix such that $\operatorname{Im} N(X) = \operatorname{Ker} X$. The transpose of X is denoted X^*. When X is a symmetric matrix, then $X > 0$ denotes that X is positive definite; likewise for symmetric matrices X and Y the notation $X > Y$ is equivalent to $X - Y > 0$. For any vector $x \in \mathbb{R}^n$, the norm $||\cdot||$ denotes the Euclidean norm $||x||_2 = \sqrt{x^*x}$. For any symmetric, positive definite matrix X the corresponding norm is denoted $||x||_X = \sqrt{x^*Xx}$.

2.1 Mathematical Analysis

Development of the results that follow requires that matrix inequalities be exchanged with equivalent inequalities of different forms. A well-known result which will be used frequently is the following, known as the Schur complement formula.

Proposition 2.1. *Consider a symmetric matrix X of the form*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix}$$

If X_3 is invertible then $X < 0$ if and only if $X_3 < 0$ and

$$X_1 - X_2 X_3^{-1} X_2 < 0$$

Likewise if X_1 is invertible then $X < 0$ if and only if $X_1 < 0$ and

$$X_3 - X_2^* X_1^{-1} X_2 < 0$$

Proof. As an intermediate result note that if for a matrix X and invertible matrix P, $X < 0$ if and only if $P^* X P < 0$. To demonstrate this, suppose that $x^* X x \leq -\epsilon ||x||$ and therefore $x^* P^* X P x \leq -\epsilon ||P||^2 ||x||$, while if there exists a y such that $y^* X y \geq 0$, then $P^{-1} y$ is such that $(P^{-1} y)^* (P^* X P)(P y) \geq 0$. To show the equivalence between the first two conditions, consider

$$\begin{bmatrix} I & -X_2 X_3^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ -X_3^{-1} X_2 & I \end{bmatrix} = \begin{bmatrix} X_1 - X_2 X_3^{-1} X_2^* & 0 \\ 0 & X_3 \end{bmatrix}$$

and so $X < 0$ if and only if each diagonal block of the right-hand side is also negative definite. The proof of the second half of the theorem proceeds in the same way. $\square$

Another important lemma for developing linear conditions is the following elimination lemma, which provides feasibility conditions for a linear inequality which no longer involve the unknown variable. The same proof can be found, for example, in [9, 7].

Proposition 2.2. *Consider matrices F, G, and symmetric matrix H. Then there exists a matrix J*

(of compatible dimension) such that

$$H + F^* J G + G^* J^* F < 0 \tag{2.1}$$

if and only if

$$N(F)^* H N(F) < 0 \tag{2.2a}$$

$$N(G)^* H N(F) < 0 \tag{2.2b}$$

Proof. The proof is by construction. Choose full-column-rank matrices V_1, V_2, and V_3 such that

$$\operatorname{Im} V_1 = \operatorname{Ker} F \cap \operatorname{Ker} G, \qquad \operatorname{Im} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \operatorname{Ker} F, \qquad \operatorname{Im} \begin{bmatrix} V_1 & V_3 \end{bmatrix} = \operatorname{Ker} G$$

Finally, select V_4 such that

$$V = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix}$$

is both square and nonsingular. Then the inequality in (2.1) holds if and only if

$$V^* H V + V^* F^* J G V + V^* G^* J^* F V < 0 \tag{2.3}$$

also holds. Partition the matrices FV and GV to conform with V, and since V forms a basis this partition has the form

$$FV = \begin{bmatrix} 0 & 0 & F_1 & F_2 \end{bmatrix}, \qquad GV = \begin{bmatrix} 0 & G_1 & 0 & G_2 \end{bmatrix}$$

Now the partition $\begin{bmatrix} F_1 & F_2 \end{bmatrix}$ has full column rank by definition, as does $\begin{bmatrix} G_1 & G_2 \end{bmatrix}$, so by defining

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} F_1^* \\ F_2^* \end{bmatrix} J \begin{bmatrix} G_1 & G_2 \end{bmatrix}$$

the matrix K can be chosen freely with an appropriate choice of J. Now define the block components of V^*HV by

$$V^*HV = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^* & H_{22} & H_{23} & H_{24} \\ H_{13}^* & H_{23}^* & H_{33} & H_{34} \\ H_{14}^* & H_{24}^* & H_{34}^* & H_{44} \end{bmatrix}$$

Using these previous definitions and inequality (2.3) yields

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^* & H_{22} & H_{23} + Y_{11}^* & H_{24} + Y_{21}^* \\ H_{13}^* & H_{23}^* + Y_{11} & H_{33} & H_{34} + Y_{12} \\ H_{14}^* & H_{24}^* + Y_{21} & H_{34}^* + Y_{12}^* & H_{44} + Y_{22} + Y_{22}^* \end{bmatrix} < 0$$

Now applying the Schur complement to the upper 3×3 block gives the following two inequalities

$$\bar{H} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^* & H_{22} & H_{23} + Y_{11}^* \\ H_{13}^* & H_{23}^* + Y_{11} & H_{33} \end{bmatrix} < 0$$

$$H_{44} + Y_{22} + Y_{22}^* - \begin{bmatrix} H_{14} \\ H_{24} + Y_{21}^* \\ H_{34} + Y_{12} \end{bmatrix}^* \bar{H}^{-1} \begin{bmatrix} H_{14} \\ H_{24} + Y_{21}^* \\ H_{34} + Y_{12} \end{bmatrix} < 0$$

Since Y is freely chosen, whenever Y_{11} is chosen to satisfy the first inequality the remaining components of Y can be chosen to satisfy the second. Now applying the Schur complement formula again with respect to H_{11} gives the inequality

$$\begin{bmatrix} H_{11} & 0 & 0 \\ 0 & H_{22} - H_{12}^* H_{11}^{-1} H_{12} & Y_{11}^* + H_{23} - H_{12}^* H_{11}^{-1} H_{13} \\ 0 & Y_{11} + H_{23}^* - H_{13}^* H_{11}^{-1} H_{12} & H_{33} - H_{13}^* H_{11}^{-1} H_{13} \end{bmatrix} < 0$$

and since Y_{11} can be chosen as needed, the inequality is satisfied if and only if the diagonal entries of this matrix are negative definite. Using Schur complements, these blocks are equivalent to

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} < 0, \qquad \begin{bmatrix} H_{11} & H_{13} \\ H_{13}^* & H_{33} \end{bmatrix} < 0$$

and from our definition of the blocks of $V^* H V$ these two inequalities are exactly those of (2.2). □

One additional technical lemma will be useful for generating linear conditions in the following chapter. It provides exact conditions under which a matrix X, partitioned appropriately, can be reconstructed from its upper left block and the upper left block of its inverse.

Proposition 2.3. *Suppose that $R, S \in \mathbb{R}^{n \times n}$ are symmetric, positive definite matrices. There exist matrices $R_2, S_2 \in \mathbb{R}^{n \times \tilde{n}}$ and symmetric matrices $R_3, S_3 \in bbr^{\tilde{n} \times \tilde{n}}$ such that*

$$\begin{bmatrix} R & R_2 \\ R_2^* & R_3 \end{bmatrix} > 0 \quad \textit{and} \quad \begin{bmatrix} R & R_2 \\ R_2^* & R_3 \end{bmatrix}^{-1} = \begin{bmatrix} S & S_2 \\ S_2^* & S_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \quad \textit{and} \quad \textit{rank} \begin{bmatrix} R & I \\ I & S \end{bmatrix} \leq n + \tilde{n}$$

Proof. First suppose that the first two conditions hold. This requires that

$$\begin{bmatrix} R & R_2 \\ R_2^* & R_3 \end{bmatrix} \begin{bmatrix} S & S_2 \\ S_2^* & S_3 \end{bmatrix} = I$$

The equations formed by carrying out the matrix multiplication on the left for each block can be used to confirm that

$$\begin{bmatrix} I & 0 \\ S & S_2 \end{bmatrix} \begin{bmatrix} R & R_2 \\ R_2^* R_3 & \end{bmatrix} \begin{bmatrix} I & S \\ 0 & S_2^* \end{bmatrix} = \begin{bmatrix} R & I \\ I & S \end{bmatrix}$$

where the left-hand side of the equality is nonnegative definite. The left-hand side also involves a matrix of dimension $n + \tilde{n}$, so the right hand side has rank of at most $n + \tilde{n}$.

Now to show necessity, suppose the latter two conditions hold. Then the Schur complement decomposition gives

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} = \begin{bmatrix} I & S^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} R - S^{-1} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & 0 \\ S^{-1} & I \end{bmatrix}$$

Since the inner term must be of rank no greater than $n + \tilde{n}$ and the left-hand side is nonnegative definite, it must be that $R - S^{-1} \geq 0$ and that $\operatorname{rank}(X - Y^{-1}) \leq \tilde{n}$. These two facts imply the existence of a matrix R_2 such that

$$R - S^{-1} = R_2 R_2^* \geq 0$$

and subtraction yields $R - R_2 R_2^* > 0$. Then from the Schur complement formula this gives

$$\begin{bmatrix} R & R_2 \\ R_2^* & I \end{bmatrix} > 0$$

and from the definition of R_2 it is easy to verify that

$$\begin{bmatrix} R & R_2 \\ R_2^* & R_3 \end{bmatrix}^{-1} = \begin{bmatrix} S & -SR_2 \\ -R_2^* S & R_2^* S R_2 + I \end{bmatrix}$$

which completes the reconstruction of the desired $X \in \mathbb{R}^{n+\tilde{n}}$. $\square$

It is worth highlighting that the rank condition of the previous proposition is not felt when $\tilde{n} \geq n$, leaving only the inequality constraint, which is a convex condition. This observation plays a role in the choice of controller rank in later developments.

2.2 Analysis of Time-Varying Systems

The focus of this thesis is on the switched linear system, which is a form of linear time-varying system. For this reason a brief treatment of LTV systems is now presented, along with a few results which will be used later. An autonomous linear time-varying system is specified by a sequence $\mathcal{G} = (A(0), A(1), \dots)$ of matrices $A(t) \in \mathbb{R}^{n \times n}$. The discrete-time linear time-varying system formed by $\mathcal{G}$ obeys the difference equation

$$x(t+1) = A(t)x(t) \tag{2.4}$$

The stability of such an autonomous system is defined in the following way.

Definition 2.4. A linear time-varying system is *uniformly exponentially stable* if there exits a $c \geq 1$ and $\lambda \in (0, 1)$ such that

$$||x(t)|| \leq c\lambda^{t-t_0}||x(t_0)|| \tag{2.5}$$

for every $t \geq t_0 \geq 0$ and every $x_(t_0) \in \mathbb{R}^n$.

This definition alone does not provide an effective way to test for the uniform exponential stability of a particular LTV system. The following result gives a characterization of uniform stability in terms of a sequence of linear matrix inequalities.

Lemma 2.5. *Suppose that the sequence $\mathcal{G}$ is bounded. Then the system $\mathcal{G}$ is uniformly exponentially stable if and only if there exist constants $\alpha, \beta, \gamma > 0$ and symmetric matrices $X(t) \in \mathbb{R}^{n\times n}$ such that for all $t \geq 0$*

$$\alpha I \leq X(t) \leq \beta I \tag{2.6a}$$

$$A(t)^* X(t+1) A(t) - X(t) \leq -\gamma I \tag{2.6b}$$

if and only if there exist constants $\delta, \epsilon, \eta > 0$ and symmetric matrices $Y(t) \in \mathbb{R}^{n\times n}$ such that for all $t \geq 0$

$$\delta I \leq Y(t) \leq \epsilon I \tag{2.7a}$$

$$A(t) Y(t) A(t)^* - Y(t+1) \leq -\eta I \tag{2.7b}$$

where the conditions in (2.6) *and* (2.7) *are related by the relationship $X(t) = Y(t)^{-1}$.*

Proof. The equivalence between uniform exponential stability and the inequalities in (2.6) are a result of the operator analysis of linear time-varying systems presented in [6]; the details are tangential to the developments of this thesis. The Schur complement theorem demonstrates the equivalence between the conditions of (2.6) and (2.7). □

An important property of the inequalities in the previous lemma is that the uniform bounds α, β, and γ exactly determine the constants c and λ required by Definition 2.4, independent of the specific sequence of $X(t)$ given.

Lemma 2.6. *Suppose there exist α, β, and γ and $X(t) \in \mathbb{R}^{n\times n}$ such that* (2.6) *holds. Then the system is uniformly exponentially stable with constants*

$$c = \sqrt{\frac{\beta}{\alpha}}, \qquad \lambda = \sqrt{1 - \frac{\gamma}{\beta}} \tag{2.8}$$

Proof. Suppose that (2.6) holds. Then the relationship $A(t)x(t) = x(t+1)$ and the inequalities in (2.6b) give

$$||x(t+1)||^2_{X(t+1)} - ||x(t)||^2_{X(t)} \leq -\gamma ||x(t)||^2$$

From (2.6a), bounds on the norm $||\cdot||_{X(t)}$ are given by

$$\alpha ||x||^2 \leq ||x||^2_{X(t)} \leq \beta ||x||^2 \tag{2.9}$$

Substitution shows that $\gamma < \beta$ and that

$$||x(t)||^2_{X(t+1)} \leq (1 - \frac{\gamma}{\beta}) ||x(t_0)||^2_{X(t)}$$

Iterating this inequality produces

$$||x(t)||^2_{X(t)} \leq (1 - \frac{\gamma}{\beta})^{t-t_0} ||x(t_0)||^2_{X(t_0)}$$

Finally, one more application of (2.9) gives

$$||x(t)||^2 \leq \frac{\beta}{\alpha}(1 - \frac{\gamma}{\beta}) ||x(t_0)||^2$$

and taking square roots completes the argument. □

Chapter 3

Stabilization of Switched Linear Systems

This chapter presents exact conditions for the existence of a uniformly stabilizing controller for a discrete-time switched linear systems using knowledge of a finite horizon of future switching modes. The existence of a uniformly stabilizing controller which a path-dependent controller has been previously characterized [14] when the future horizon length is zero (controllers which also achieve specified disturbance attenuation are considered in [13]). The results in this chapter serve as extensions of this work to the case where the controller has access to a finite future horizon as well.

The developments in this chapter are organized as follows: first, the uniform stability of an autonomous switched system is explored and characterized by a finite family of linear matrix inequalities. Next, the stabilization of a switched linear system using a full-information feedback controller is examined and linear matrix inequalities are developed which characterize the existence of such a controller. Finally, uniform stabilizaton via dynamic output-feedback controllers is considered and conditions for a uniformly stabilizing controller are presented in the form of two groups of linear matrix inequalities along with a coupling condition.

3.1 Stability of Autonomous Systems

The stability of an autonomous system serves as the foundation for the developments that follow. Once a controlled system is combined with a particular controller, the resulting closed-loop system will behave as an autonomous system. It is therefore necessary to characterize the stability of such a system first. Consider a finite set

$$\mathcal{G} = \{A_0, A_1, \ldots, A_N\} \tag{3.1}$$

where each $A \in \mathbb{R}^{n \times n}$. This set contains the parameters among which the system will switch. Next, let $Q \in \{0,1\}^{N \times N}$ be a row-allowable matrix. The switching sequence which governs this system is required to satisfy the constraint that $\theta(t) = i$ and $\theta(t+1) = j$ only if the (i,j) entry of Q is nonzero. This is equivalent to requiring that $\theta(\cdot)$ be a valid walk on the directed graph given by Q. The following definition formalizes this notion.

Definition 3.1. We call a switching sequence $\theta(\cdot)$ *admissible* if for each $t \geq 0$ and $i, j \in \{1, \ldots, N\}$ we have $\theta(t) = i$ and $\theta(t+1) = j$ only if the (i,j) entry of Q is nonzero.

The parameter set $\mathcal{G}$ and the switching constraint Q together define the switched linear system $(\mathcal{G}, Q)$, which is governed by the difference equation

$$x(t+1) = A_{\theta(t)} x(t) \tag{3.2}$$

For a fixed switching sequence $\theta(\cdot)$, this is simply a linear time-varying system for which exponential stability is given in Definition 2.4. However the goal of the controller stabilization that follows is to provide uniform stabilization over all allowable switching sequences as well as time.

Definition 3.2. A switched linear system $(\mathcal{G}, Q)$ is *uniformly exponentially stable* if there exists a $c \geq 1$ and $\lambda \in (0,1)$ such that

$$||x(t)|| \leq c\lambda^{t-t_0} ||x(t_0)|| \tag{3.3}$$

for every $t \geq t_0 \geq 0$, $x(t_0) \in \mathbb{R}^n$ and every admissible switching sequence.

There are, in general, infinitely many switching sequences to consider and so checking each for uniform stability is not feasible. The stabilization result that follows demonstrates the existence of a quadratic Lyapunov function which depends on a finite-length switching path, comprised of a finite path memory and a finite future horizon. This results in conditions for the stability of the system in the form of a finite collection of linear matrix inequalities.

The statement of the following theorem makes use of a function σ which assigns to each finite-length path a corresponding parameter from our set $\mathcal{G}$. Such a function can be thought of as selecting the mode corresponding to the present from a path which contains knowledge of both the past and future modes. In order to show these conditions are necessary as well as sufficient, the switching constraint Q is required to be *strongly connected*; that is, for any $i, j \in \{1, \ldots, N\}$ there is an admissible path in Q beginning at i and ending at j.

Theorem 3.3. *Consider the system given in* (3.2) *and let* $H \geq 0$ *be the length of the future horizon. Also, let* Q *be strongly connected. Then the system is uniformly exponentially stable if and only if there exists an integer* $M \geq 0$*, a collection of positive definite matrices* $X_{j_1,\ldots,j_{M+H}}$ *and a function* $\sigma : \{1, \ldots, N\}^{H+1} \mapsto \{1, \ldots, N\}$ *such that*

$$A^*_{\sigma(j_M,\ldots,j_{M+H})} X_{j_1,\ldots,j_{M+H}} A_{\sigma(j_M,\ldots,j_{M+H})} - X_{j_0,\ldots,j_{M+H-1}} < 0 \tag{3.4}$$

for every admissible switching path $(j_0, \ldots, j_{M+H}) \in \{1, \ldots, N\}^{M+H+1}$.

Proof. To show sufficiency, suppose there exists a collection of positive definite matrices $X_{j_1,\ldots,j_{M+H}}$ which satisfy the inequalities in (3.4). Since there are only finitely many paths of length $M+H+1$ (and therefore only finitely many inequalities), constants $\alpha, \beta, \gamma > 0$ can be chosen to be uniform bounds such that

$$\alpha I \leq X_{j_1,\ldots,j_{M+H}} \leq \beta I$$

$$A^*_{\sigma(j_M,\ldots,j_{M+H})} X_{j_1,\ldots,j_{M+H}} A_{\sigma(j_M,\ldots,j_{M+H})} - X_{j_0,\ldots,j_{M+H-1}} < -\gamma I$$

Let $\theta(\cdot)$ be any admissible switching sequence defined for $t \geq 0$. Extend this sequence to the left as follows: choose modes $i_{-M}, \ldots, i_{-1}$ such that the entries $(i_{-M}, i_{-M+1}), \ldots, (i_{-1}, \theta(0))$ of Q are all nonzero. This makes $(i_{-M}, \ldots, i_{-1}, \theta(0)\}$ an admissible switching sequence. Define a sequence $X(\cdot)$ by using:

$$X(t) = \begin{cases} X_{i_{-M},\ldots,i_{-1}} & t = 0 \\ X_{i_{t-M},\ldots,i_{-1},\theta(0),\ldots\theta(t-1)} & 0 < t < M \\ X_{\theta(t-M),\ldots,\theta(t-1)} & t \geq M \end{cases} \tag{3.5}$$

and define $A(t) = A_{\sigma(\theta(t),\ldots,\theta(t+H))}$ for $t \geq 0$. These definitions and the inequalities in (3.4) provide the following inequalities for every $t \geq 0$

$$\alpha I \leq X(t) \leq \beta I$$

$$A(t)^* X(t+1) A(t) - X(t) < -\gamma I$$

From the result of Lemmas 2.5 and 2.6, this implies that the system is uniformly exponentially stable and further that the constants c, λ can be chosen as functions solely of α, β, and γ, which are independent of the switching sequence chosen. Since the switching sequence chosen was arbitrary, these constants hold uniformly over all admissible sequences.

To show necessity, suppose the system in (3.2) is uniformly exponentially stable. For any admissible switching sequence define the operator

$$A_\theta = \begin{bmatrix} A_{\theta(0)} & 0 & \ldots \\ 0 & A_{\theta(1)} & \\ \vdots & & \ddots \end{bmatrix}$$

and also define the unilateral shift operator $Z : (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots)$. This system then takes the form

$$x = ZA_\theta x$$

Using the operator approach of [6], the uniform exponential stability of this time-varying system means there exist constants $c \geq 1$ and $\lambda \in (0, 1)$ such that

$$||(ZA_\theta)^t||_{l2 \mapsto l2} \leq c\lambda^t \tag{3.6}$$

Now consider the Lyapunov equation for this system given by

$$(ZA_\theta)Y(ZA_\theta)^* - Y = -I \tag{3.7}$$

Whenever $\rho(ZA_\theta) < 1$ the solution to this equation is given by

$$Y = \sum_{k=0}^{\infty} (ZA_\theta)^k \left[(ZA_\theta)^*\right]^k$$

Notice that each term in the summation has a block-diagonal structure, and therefore the solution Y will also be block-diagonal. Choose an $M > 0$ such that $c\lambda^{M+1} < 1$. Using this define

$$Y^{(M)} = \sum_{k=0}^{M} (ZA_\theta)^k \left[(ZA_\theta)^*\right]^k$$

and also

$$E^{(M)} = \sum_{k=M+1}^{\infty} (ZA_\theta)^k \left[(ZA_\theta)^*\right]^k$$

such that $Y = Y^{(M)} + E^{(M)}$. Now it follows from the definition of $E^{(M)}$ that

$$E^{(M)} - (ZA_\theta)E^{(M)}(ZA_\theta^* = (ZA_\theta)^{M+1} \left[(ZA_\theta)^{M+1}\right]^* < c^2\lambda^{2(M+1)} I \tag{3.8}$$

where the final inequality comes from norm bound given in (3.6). Substitute $Y = Y^{(M)} + E^{(M)}$ into (3.7), rearrange terms, and notice that $c^2\lambda^{2(M+1)} < 1$ to find

$$\begin{aligned}(ZA_\theta)Y^{(M)}(ZA_\theta)^* - Y^{(M)} &= -I + \left(E^{(M)} - (ZA_\theta)E^{(M)}(ZA_\theta)^*\right) \\ &< -(1 - c^2\lambda^{2(M+1)})I < 0\end{aligned} \tag{3.9}$$

and therefore $Y^{(M)}$ satisfies the Lyapunov inequality. Examining the block structure of $Y^{(M)}$ shows

$$(Y^{(M)})_t = I + \sum_{s=t-M}^{t-1} (A_{\theta(s)} \dots A_{\theta(t-1)})(A_{\theta(s)} \dots A_{\theta(t-1)})^*$$

Introduce the function σ such that $\sigma(\theta(s), \dots, \theta(s+H)) = \theta(s)$ for every $s \geq 0$; then the above summation can be rewritten as

$$(Y^{(M)})_t = I + \sum_{s=t-M}^{t-1} (\Phi(s, t-1)))(\Phi(s, t-1))^*$$

where $\Phi(s, t-1) = A_{\sigma(\theta(s),\dots,\theta(s+H))} \dots A_{\sigma(\theta(t-1),\dots,\theta(t+H-1))}$. Now the identity term in this block guarantees that $(Y^{(M)})_t$ is positive definite, and each term in the summation is exactly determined by the switching sequence $(\theta(t-M), \dots, \theta(t+H))$. Therefore denote $(Y^{(M)})_t = Y_{\theta(t-M),\dots,\theta(t+H-1)}$. Examining the $t+1$ block of the Lyapunov inequality in (3.9) (and noting that $(ZXZ^*)_{t+1} = (X)_t$), provides the sequence of inequalities

$$A_{\sigma(\theta(t),\dots,\theta(t+H))} Y^{(M)}_{\theta(t-M),\dots,\theta(t-1)} A^*_{\sigma(\theta(t),\dots,\theta(t+H))} - Y^{(M)}_{\theta(t-M+1),\dots,\theta(t)} < 0$$

Applying the Schur complement formula twice and defining $X(t) = Y(t)^{-1}$ gives the new sequence of inequalities

$$A^*_{\sigma(\theta(t),\dots,\theta(t+H))} X^{(M)}_{\theta(t-M+1),\dots,\theta(t)} A_{\sigma(\theta(t),\dots,\theta(t+H))} - X^{(M)}_{\theta(t-M),\dots,\theta(t+H-1)} < 0 \tag{3.10}$$

The above sequence of inequalities holds for any switching sequence by uniform stability. Because the switching constraint graph is strongly connected, there exists a recurrent switching sequence $\theta(\cdot)$; i.e. a sequence in which every admissible sequence of length $M + H + 1$ appears infinitely often. Then by considering the sequence of inequalities (3.10) starting from any time $t \geq M$, every single inequality required by (3.4) appears (infinitely often), demonstrating necessity.

□

As a brief aside, note that the set of inequalities described by (3.4) can grow combinatorially as M grows large, and with it the number of positive definite matrices to be found. It is possible that a more sparse solution exists (i.e. the $X_{j_1,\ldots,j_{M+H}}$ are not distinct for every admissible path).

Corollary 3.4. *Consider the system given in* (3.2) *and let* $H \geq 0$*. Also let* Q *be strongly connected. Then the system is uniformly exponentially stable if and only if there exists an integer* $M \geq 0$*, an integer* $J > 0$*, a collection of positive definite matrices* $X_1, \ldots, X_J$*, a mapping* $\phi : (j_1, \ldots, j_{M+H}) \mapsto \{1, \ldots, J\}$ *and a function* $\sigma : \{1, \ldots, N\}^{H+1} \mapsto \{1, \ldots, N\}$ *such that*

$$A^*_{\sigma(j_M,\ldots,j_{M+H})} X_{\phi(j_1,\ldots,j_{M+H})} A_{\sigma(j_M,\ldots,j_{M+H})} - X_{\phi(j_0,\ldots,j_{M+H-1})} < 0 \tag{3.11}$$

for every admissible switching path $(j_0, \ldots, j_{M+H}) \in \{1, \ldots, N\}$.

Proof. The proof of sufficiency follows nearly identically the proof of sufficiency in Theorem 3.3, where the definition of $X(\cdot)$ in (3.5) is modified to be

$$X(t) = \begin{cases} X_{\phi(i_{-M},\ldots,i_{-1})} & t = 0 \\ X_{\phi(i_{t-M},\ldots,i_{-1},\theta(0),\ldots\theta(t-1))} & 0 < t < M \\ X_{\phi(\theta(t-M),\ldots,\theta(t-1))} & t \geq M \end{cases} \tag{3.12}$$

The proof of necessity is exactly that of Theorem 3.3, where the required mapping ϕ can be taken as an enumeration of the paths of length $M + H$. □

3.2 Stabilization Using State Feedback

The existence of a uniformly stabilizing controller for a switched linear system can now be examined. This section develops conditions for the existence of a full-information state feedback controller, while the existence of a dynamic output-feedback controller is considered in the following section. When a controller is specified and connected in feedback with the system, the resulting closed-loop system can be treated as autonomous and the uniform stability of the system is given by the conditions in Theorem 3.3. The resulting inequalities are not simultaneously linear in both the controller parameters and the $X_{j_1,\ldots,j_{M+H}}$, so additional work is required to produce linear inequalities. The method used is based on the results in [9, 17] (the same method is presented in [7] for continuous LTI systems and is applied to switched linear systems in [14, 13]).

Consider the finite set

$$\mathcal{G} = \{(A_1, B_1), \ldots, (A_N, B_N)\} \tag{3.13}$$

where each ordered pair represents the parameters of one of the system modes, with $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$. Again, let $Q \in \{0,1\}^{N\times N}$ be a row-allowable matrix which determines the allowable switching sequences for the system. The (controlled) switched linear system $((G), Q)$ is then represented as

$$x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) \tag{3.14}$$

for some admissible switching sequence $\theta(\cdot)$. This system is to be connected in state feedback with a control law $\mathcal{K}$ of the form

$$u(t) = K(t)x(t) \tag{3.15}$$

such that the resulting closed-loop system formed from (3.14) is uniformly exponentially stable. The full state of the system is available to the controller; perfect knowledge of the switching signal up to a finite future horizon of length $H \geq 0$ is also permitted.

Substitution of the control law (3.15) into the system's governing equations in (3.14) produce the closed loop system $(\mathcal{G}, Q, \mathcal{K})$ specified by

$$x(t+1) = \hat{A}(t)x(t) \tag{3.16}$$

where the new system variable $\hat{A}(t)$ is defined as

$$\hat{A}(t) = A_{\theta(t)} + B_{\theta(t)}K(t) \tag{3.17}$$

When the control law at each time is chosen based on a finite-length memory of past states and a finite future horizon, the resulting closed-loop system variables $\hat{A}$ form a finite set governed by a switching law. The uniform stability of this system is exactly characterized by Theorem 3.3, as the following lemma shows.

Lemma 3.5. *Consider the system in* (3.16)*; let $\mathcal{K}$ be finite-path dependent with future horizon $H \geq 0$, and let Q be strongly connected. The closed-loop system $(\mathcal{G}, Q, \mathcal{K})$ is uniformly exponentially stable if and only if there exists an integer $M > 0$ and a collection of positive definite matrices $X_{j_1,\ldots,j_{M+H}} \in \mathbb{R}^{n\times n}$ such that*

$$\hat{A}^*(t)X(t+1)\hat{A}(t) - X(t) < 0 \tag{3.18}$$

for all $t \geq 0$ where

$$X(t) = \begin{cases} X_{i_{-M},\ldots,i_{-1}} & t = 0 \\ X_{i_{t-M},\ldots,i_{-1},\theta(0),\ldots\theta(t-1)} & 0 < t < M \\ X_{\theta(t-M),\ldots,\theta(t-1)} & t \geq M \end{cases} \tag{3.19}$$

Proof. The proof of this lemma follows almost immediately from that of Theorem 3.3 to the (autonomous) closed-loop system. Sufficiency is obtained directly; for necessity, suppose that the system $(\mathcal{G}, Q, \mathcal{K})$ is uniformly exponentially stable. If the controller is dependent on a past of length L and M is a nonnegative integer such that $c\lambda^{M+1} < 1$, then following the proof of necessity using the constant $\hat{M} = M + l$ achieves the desired result. $\square$

The family of Lyapunov inequalities given by this lemma are not simultaneously linear in both the controller variable K and the $X_{j_1,\ldots,j_{M+H}}$, so additional work is needed in order to produce a family of inequalities which are linear. Applying the Schur complement formula to (3.18) yields the following inequality

$$\tilde{H}_{\theta(t-M)\ldots,\theta(t+H)} + \tilde{G}^*_{\theta(t)} K(t)^* \tilde{F}_{\theta(t)} + \tilde{F}^*_{\theta(t)} K(t) \tilde{G}_{\theta(t)} \tag{3.20}$$

where

$$\tilde{F}_{\theta(t)} = \begin{bmatrix} B^*_{\theta(t)} & 0 \end{bmatrix} \qquad \tilde{G}_{\theta(t)} = \begin{bmatrix} 0 & I \end{bmatrix}$$

$$\tilde{H}_{\theta(t-M)\ldots,\theta(t+H)} = \begin{bmatrix} -X^{-1}_{\theta(t-M+1),\ldots,\theta(t+H)} & A_{\theta(t)} \\ A^*_{\theta(t)} & -X_{\theta(t-M),\ldots,\theta(t+H-1)} \end{bmatrix}$$

These inequalities are linear in $K(t)$, and applying Proposition 2.2 gives conditions for when they are feasible in $K(t)$.

Lemma 3.6. *There exists a matrix $K(t)$ which satisfies the inequality* (3.20) *if and only if*

$$N(\tilde{F}_{\theta(t)})^* \tilde{H}_{\theta(t-M)\ldots,\theta(t+H)} N(\tilde{F}_{\theta(t)}) < 0 \tag{3.21a}$$

$$N(\tilde{G}_{\theta(t)})^* \tilde{H}_{\theta(t-M)\ldots,\theta(t+H)} N(\tilde{G}_{\theta(t)}) < 0 \tag{3.21b}$$

Now these inequalities are still not linear as both $X_{\theta(t-M+1),\ldots,\theta(T+H)}$ and $X^{-1}_{\theta(t-M+1),\ldots,\theta(T+H)}$ appear within $H_{\theta(t-M)\ldots,\theta(t+H)}$. The following theorem gives equivalent matrix inequalities which are linear in all variables.

Theorem 3.7. *The switched linear system described by* (3.14) *is stabilizable by a finite-path dependent controller with future horizon $H \geq 0$ if and only if there exist an integer $M \geq 0$ and* *symmetric positive definite matrices* $R_{j_1,\ldots,j_{M+H}} \in \mathbb{R}^{n\times n}$ *such that*

$$N(B_{j_M})^*(A_{j_M}R_{j_0,\ldots,j_{M+H-1}}A_{j_M}^* - R_{j_1,\ldots,j_{M+H}})N(B_{j_M}) < 0 \tag{3.22}$$

Proof. An explicit construction of the full-rank matrix $N(F_{\theta(t)}$ is given by

$$N(\tilde{F}_{\theta(t)}) = \begin{bmatrix} N(B_{\theta(t)}^*) & 0 \\ 0 & I \end{bmatrix}$$

Substitution of this representation into (3.21a) gives

$$\begin{bmatrix} N(B_{\theta(t)^*})^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -X_{\theta(t-M+1),\ldots,\theta(t+H)}^{-1} & A_{\theta(t)} \\ A_{\theta(t)}^* & -X_{\theta(t-M),\ldots,\theta(t+H-1)} \end{bmatrix} \begin{bmatrix} N(B_{\theta(t)^*}) & 0 \\ 0 & I \end{bmatrix} < 0$$

Carrying out the matrix multiplication and applying the Schur complement formula produces

$$N(B_{\theta(t)}^*)^*(A_{\theta(t)}X_{\theta(t-M),\ldots,\theta(t+H-1)}^{-1}A_{\theta(t)}^* - X_{\theta(t-M+1),\ldots,\theta(t+H)}^{-1})N(B_{\theta(t)}^*) < 0$$

from which the change of variable $R_{j_1,\ldots,j_{M+H}} = X_{j_1,\ldots,j_{M+H}}^{-1}$ gives the inequality in (3.22). Substitution of a representation of $N(\tilde{G}_{\theta(t)})$ into (3.21b) yields $X_{j_1,\ldots,j_{M+H}}^{-1} > 0$, which is trivially satisfied. □

Once a feasible family of inequalities is solved for a particular M, controller synthesis proceeds by substituting the $X_{j_1,\ldots,j_{M+H}} = R_{j_1,\ldots,j_{M+H}}^{-1}$ back into the inequalities in (3.20). These inequalities can then be solved to find controller gains $K_{j_0,\ldots,j_{M+H}}$ for each feasible path. The controller is then implemented as

$$K(t) = \begin{cases} K_{j_{-M},\ldots,j_{-1},\theta(0),\ldots,\theta(H)} & t = 0 \\ K_{j_{t-M},\ldots,j_{-1}}, \theta(0), \ldots, \theta(t+H) & 0 \leq t < M \\ K_{\theta(t-M),\ldots,\theta(t+H)} & t \geq M \end{cases} \tag{3.23}$$

3.3 Stabilization Using Output Feedback

Using the same techniques as in the case of a full-information state feedback controller, conditions for the existence of an output-feedback controller can now be developed. Take the finite set of system parameters given by

$$\mathcal{G} = \{(A_1, B_1, C_1), \ldots, (A_N, B_N, C_N)\} \tag{3.24}$$

in which every $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, and $C_i \in \mathbb{R}^{p \times n}$. Also, let $Q \in \{0, 1\}^{N \times N}$ be a row-allowable matrix representing the switching constraint for the system. The resulting switched system $(\mathcal{G}, Q)$ has the form

$$x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) \tag{3.25a}$$

$$y(t) = C_{\theta(t)}x(t) \tag{3.25b}$$

The system is connected in feedback with a controller $\mathcal{K}$ of the form

$$x_K(t+1) = A_K(t)x_K(t) + B_K(t)y(t) \tag{3.26a}$$

$$u(t) = C_K(t)x_K(t) + D_K(t)y(t) \tag{3.26b}$$

in which $A_K(t) \in \mathbb{R}^{n_K \times n_K}$, $B_K(t) \in \mathbb{R}^{n_K \times p}$, $C_K(t) \in \mathbb{R}^{m \times n_K}$, and $D_K(t) \in \mathbb{R}^{m \times p}$. As in the full-information case, the finite-path dependent controller with future horizon $H \geq 0$ is desired.

Defining the following matrices

$$\hat{A}(t) = \begin{bmatrix} A_{\theta(t)} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{B}(t) = \begin{bmatrix} 0 & B_{\theta(t)} \\ I & 0 \end{bmatrix}, \qquad \hat{C}(t) = \begin{bmatrix} 0 & I \\ C_{\theta(t)} & 0 \end{bmatrix} \tag{3.27}$$

and also

$$K(t) = \begin{bmatrix} A_K(t) & B_K(t) \\ C_K(t) & D_K(t) \end{bmatrix} \tag{3.28}$$

and using $x_C(t) = \begin{bmatrix} x^*(t) & x_K^*(t) \end{bmatrix}^*$ as the state, the closed-loop system $(\mathcal{G}, Q, \mathcal{K})$ takes the form

$$x_C(t+1) = A_C(t) x_C(t) \tag{3.29}$$

where the matrix A_C is defined by

$$A_C(t) = \hat{A}(t) + \hat{B}(t) K(t) \hat{C}(t) \tag{3.30}$$

Conditions for the uniform stability of this closed-loop system can now be stated in nearly the same form as for the full-information case

Lemma 3.8. *Consider the system in* (3.25)*; let $\mathcal{K}$ be finite-path dependent with future horizon $H \geq 0$, and let Q be strongly connected. The closed-loop system $(\mathcal{G}, Q, \mathcal{K})$ is uniformly exponentially stable if and only if there exists an integer $M \geq 0$ and a collection of positive definite matrices $X_{j_1,\ldots,j_{M+H}} \in \mathbb{R}^{n+n_K \times n+n_K}$ such that*

$$A_C^*(t) X(t+1) A_C(t) - X(t) < 0 \tag{3.31}$$

for all $t \geq 0$ *where*

$$X(t) = \begin{cases} X_{i_{-M},\ldots,i_{-1}} & t = 0 \\ X_{i_{t-M},\ldots,i_{-1},\theta(0),\ldots\theta(t-1)} & 0 < t < M \\ X_{\theta(t-M),\ldots,\theta(t-1)} & t \geq M \end{cases} \tag{3.32}$$

Proof. The proof of this lemma, like that of Lemma 3.5, follows that of Theorem 3.3. Sufficiency follows the proof exactly, while necessity comes from the choice of $\hat{M} = M + L$ as was done in the previous section. □

As before, the inequalities generated by this lemma are not linear in both the controller variable K and the $X_{j_1,\ldots,j_{M+H}}$. Applying the Schur complement allows the inequalities to be rewritten as

$$H_{\theta(t-M)\ldots,\theta(t+H)} + G^*_{\theta(t)}K(t)^*F_{\theta(t)} + F^*_{\theta(t)}K(t)G_{\theta(t)} \tag{3.33}$$

in which the following definitions are used

$$F_{\theta(t)} = \begin{bmatrix} \hat{B}^*(t) & 0 \end{bmatrix} \qquad G_{\theta(t)} = \begin{bmatrix} 0 & \hat{C}(t) \end{bmatrix}$$

$$H_{\theta(t-M)\ldots,\theta(t+H)} = \begin{bmatrix} -X^{-1}_{\theta(t-M+1),\ldots,\theta(t+H)} & \hat{A}(t) \\ \hat{A}^*(t) & -X_{\theta(t-M),\ldots,\theta(t+H-1)} \end{bmatrix}$$

Elimination of the controller variable $K(t)$ proceeds using Proposition 2.2 as before.

Lemma 3.9. *There exists a matrix* $K(t)$ *which satisfies the inequality* (3.20) *if and only if*

$$N(F_{\theta(t)})^* H_{\theta(t-M)\ldots,\theta(t+H)} N(F_{\theta(t)}) < 0 \tag{3.34a}$$

$$N(G_{\theta(t)})^* H_{\theta(t-M)\ldots,\theta(t+H)} N(G_{\theta(t)}) < 0 \tag{3.34b}$$

From these inequalities the final result for the existence of a uniformly stabilizing controller can be stated.

Theorem 3.10. *The switched linear system described by* (3.25) *is uniformly stabilizable by a finite-path dependent controller with look-ahead horizon $H \geq 0$ if and only if there exist an integer $M \geq 0$ and positive definite matrices $R_{j_1,\dots,j_{M+H}}$, $S_{j_1,\dots,j_{M+H}} \in \mathbb{R}^{n\times n}$ such that*

$$N(B_{j_M}^*)^*(A_{j_M}R_{j_0,\dots,j_{M+H-1}}A_{j_M}^* - R_{j_1,\dots,j_{M+H}})N(B_{j_M}^*) < 0 \tag{3.35a}$$

$$N(C_{j_M})^*(A_{j_M}^*S_{j_1,\dots,j_{M+H}}A_{j_M} - S_{j_0,\dots,j_{M+H-1}})N(C_{j_M}) < 0 \tag{3.35b}$$

$$\begin{bmatrix} R_{j_1,\dots j_{M+H}} & I \\ I & S_{j_1,\dots,j_{M+H}} \end{bmatrix} \geq 0 \tag{3.35c}$$

Proof. Partition the matrices $X_{j_1,\dots,j_{M+H}}$ and their inverses such that

$$X_{j_1,\dots,j_{M+H}}^{-1} = \begin{bmatrix} R_{j_1,\dots,j_{M+H}} & T_{j_1,\dots,j_{M+H}} \\ T_{j_1,\dots,j_{M+H}}^* & V_{j_1,\dots,j_{M+H}} \end{bmatrix} = \begin{bmatrix} S_{j_1,\dots,j_{M+H}} & U_{j_1,\dots,j_{M+H}} \\ U_{j_1,\dots,j_{M+H}}^* & W_{j_1,\dots,j_{M+H}} \end{bmatrix}^{-1} = X_{j_1,\dots,j_{M+H}}$$

and from Proposition 2.3 the matrices $X_{j_1,\dots,j_{M+H}}$ can be reconstructed from $R_{j_1,\dots,j_{M+H}}$ and $S_{j_1,\dots,j_{M+H}}$ if and only if the condition in (3.35c) is satisfied. Now a full-rank representation of the matrix $N(F_{\theta(t)})$ is given by

$$N(F_{\theta(t)}) = \begin{bmatrix} N(B_{\theta(t)}^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and so substitution into (3.34a) gives

$$\begin{bmatrix} N(B_{\theta(t)}^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^* \begin{bmatrix} -X_{\theta(t-M+1),\dots,\theta(t+H)}^{-1} & \hat{A}(t) \\ \hat{A}^*(t) & -X_{\theta(t-M),\dots,\theta(t+H-1)} \end{bmatrix} \begin{bmatrix} N(B_{\theta(t)}^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

Carrying out the matrix multiplication and using the partition of $X^{-1}_{\theta(t-M+1),\ldots,\theta(t+H)}$ as described results in the inequality

$$\begin{bmatrix} -N(B^*_{\theta(t)})^* R_{\theta(t-M+1),\ldots,\theta(t+H)} N(B^*_{\theta(t)}) & N(B^*_{\theta(t)})^* A_{\theta(t)} & 0 \\ A^*_{\theta(t)} N(B^*_{\theta(t)}) & & \\ & & -X_{\theta(t-M),\ldots,\theta(t+H-1)} \\ 0 & & \end{bmatrix} < 0$$

The Schur complement and the partition of $X^{-1}_{\theta(t-M),\ldots,\theta(t+H-1)}$ produces (3.35a). An identical manipulation of (3.34b) using a full-rank representation of $N(G_{\theta(t)})$ produces (3.35b). $\square$

If either B^*_i or C_i are of full column rank, then the inequalities involving $N(B^*_i)$ (or $N(C_i)$) are trivially satisfied. In particular, when every C_i is of full rank (such as the full-information case when C_i is taken to be identity), only the first inequality of the theorem remains; this is exactly the result of the full-information development in Theorem 3.7.

Proposition 2.3 requires that $R_{j_1,\ldots,j_{M+H}}$ and $S_{j_1,\ldots,j_{M+H}}$ satisfy the additional constraint

$$\operatorname{rank} \begin{bmatrix} R_{j_1,\ldots j_{M+H}} & I \\ I & S_{j_1,\ldots,j_{M+H}} \end{bmatrix} \leq n + n_K \tag{3.36}$$

As was noted previously, this constraint is not felt if $n_K \geq n$. Since the controller variables appear nowhere in (3.22), their feasibility always allows for the construction of a controller of the same order as the plant. If a reduced order controller is desired, then this rank condition must be taken into consideration. This extra condition is nonconvex and in general difficult to apply, making the problem of finding a reduced order controller harder than that of a full order controller.

A final observation can be made about the conditions developed in Theorem 3.10. As the future horizon H is increased, the length of the path $j_1, \ldots, j_{M+H}$ must also increase and with it the number of admissible paths that must be considered. This leads to a larger family of inequalities which must be solved to demonstrate the existence of a controller. Nevertheless this larger family of inequalities is not harder to satisfy, as the following proposition shows.

Theorem 3.11. *Consider the family of inequalities described by Theorem 3.10. If for a particular* M*,*H *a solution exists to this family of inequalities, then a solution exists for the family generated by increasing the future horizon.*

Proof. Suppose there exists positive definite matrices $R_{j_1,\ldots,j_{M+H}}$ and $S_{j_1,\ldots,j_{M+H}}$ which satisfy the necessary inequalities. When the future horizon is increased to $H+1$, the new matrices $\tilde{R}_{j_1,\ldots,j_{M+H+1}}$ and $\tilde{S}_{j_1,\ldots,j_{M+H+1}}$ can be chosen according to the rule

$$\tilde{R}_{j_1,\ldots,j_{M+H+1}} = R_{j_1,\ldots,j_{M+H}}, \qquad \tilde{S}_{j_1,\ldots,j_{M+H+1}} = S_{j_1,\ldots,j_{M+H}}$$

for every allowable path of length $M+H+1$. Each required inequality for horizon length $H+1$ reduces to one of the previously solved inequalities for horizon length H. $\square$

Chapter 4

Algorithms and Examples

In the previous chapter exact conditions were given for the uniform stabilization of a switched linear system. The systematic nature of constructing these conditions lends itself to a systematic search for stabilizing controllers. Algorithms for searching for such a controller are presented here. In addition, the inequalities produces for a finite horizon of length $H+1$ were shown to be feasible whenever the corresponding conditions for horizon length H are feasible. The converse is not true in general; for certain types of systems stabilization is not possible without a finite horizon of switching modes. Examples of these systems are provided along with

4.1 Algorithms for Controller Design

The conditions presented in Theorem 3.10 provide a straightforward algorithm to search for a uniformly stabilizing controller. When the system $(\mathcal{G}, Q)$ is uniformly stabilizable, the algorithm must terminate for a finite M:

1. Fix a finite look-ahead H and set $M = 0$.
2. Solve the feasibility problem given by the conditions in (3.35) for every admissible path $(j_0, \ldots, j_{M+H})$.

3. If the conditions are not feasible, increment M and return to Step 2.

4. Reconstruct the matrices $X_{j_1,\ldots,j_{M+H}}$ from $R_{j_1,\ldots,j_{M+H}}$ and $S_{j_1,\ldots,j_{M+H}}$ as described in Proposition 2.3.

5. Use the reconstructed $X_{j_1,\ldots,j_{M+H}}$ for the inequalities in (3.33) and solve for the corresponding controller variables $K_{j_0,\ldots,j_{M+H}}$.

The previous algorithm is not guaranteed to terminate if the system is not known to be uniformly stabilizable, so the following modified algorithm (specified in [14]) can be used to detect when a system may not be stabilizable.

1. Fix a finite look-ahead H and set $M = 0$, $\gamma_{-1} = \infty$.

2. Solve the modified semidefinite programming problem

$$\text{minimize } \gamma \text{ such that}$$
$$N(B^*_{j_M})^*(A_{j_M} R_{j_0,\ldots,j_{M+H-1}} A^*_{j_M} - R_{j_1,\ldots,j_{M+H}}) N(B^*_{j_M}) < \gamma I$$
$$N(C_{j_M})^*(A^*_{j_M} S_{j_1,\ldots,j_{M+H}} A_{j_M} - S_{j_0,\ldots,j_{M+H-1}}) N(C_{j_M}) < \gamma I$$
$$\begin{bmatrix} R_{j_1,\ldots j_{M+H}} & I \\ I & S_{j_1,\ldots,j_{M+H}} \end{bmatrix} \geq -gamma * I R_{j_1,\ldots,j_{M+H}} > 0, \qquad S_{j_1,\ldots,j_{M+H}}$$

   and denote the optimal value γ_M.

3. If $\gamma_M < 0$, then proceed with controller synthesis as in the previous algorithm.

4. If $\gamma_M - \gamma M - 1 < \epsilon$ for some tolerance ϵ, stop and declare the system not stabilizable.

5. Otherwise increment M and return to Step 2.

The sequence γ_M will always be nonincreasing and γ_0 must be finite (as a γ can be found which bounds the above inequalities for any finite set of positive definite matrices), so this algorithm must terminate in finite time.

Both the dimension of the system under consideration and the number of admissible paths of length $M + H + 1$ directly influence the scale of the semidefinite programming problem which must be solved in Step 2 of this algorithm. In particular the number of inequalities which must be considered grows combinatorially as M (or H) is increased. Such growth presents challenges in the formulation of the problem and in checking for a feasible solution.

A number of software packages are already available for semidefinite programming, as well as modeling languages such as `CVX` [10] which permit the compact description of a semidefinite programming problem. However, no comprehensive software package yet exists for the systematic generation of the inequalities required by (3.35). A related opportunity comes from the use of distributed solver software, allowing very large-scale problems (i.e. those with many switching modes, large-dimension models or long path dependance) to be solved more efficiently (see [2, 3] for one such distributed implementation). The development of a software toolbox for the systematic search and design of finite-path controllers is left for future work.

4.2 Numerical Examples

A simple example of switched systems which are stabilizable only through the use of a future horizon are those systems with input delays. Such systems require that a choice of control effort be made in advance of its effect on system dynamics. A change in the parameters of the system can then cause the input chosen to destabilize the system rather than stabilize it.

Example 4.1. Consider the discrete linear inclusion specified by

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A switching sequence exists which grows the state of this system without bound for any controller which depends only on the current and past modes. Indeed, suppose that $u(t)$ is generated by any feedback control law which depends on the switching modes $\theta(t), \theta(t-1), \ldots$. Without loss of generality suppose that $x_2(t_0) > 0$. Since $x(t+1) = u(t)$ for each $t \geq 0$ and the switching mode $\theta(t+1)$ is given after the choice of $u(t)$, then the switching rule

$$\theta(t) = \begin{cases} 1, & u(t-1) \geq 0 \\ 2, & u(t-1) < 0 \end{cases}$$

ensures that $x_2(t+1) \geq x_2(t)$ for every $t \geq 1$.

In contrast, when a future horizon is available the control effort can be chosen ”in advance” to anticipate the state of the system after passing through the delay. Applying the algorithm in Section 4.1 results in a controller which depends on the switching path $(\theta(t), \theta(t+1))$:

$$K_{11} = \begin{bmatrix} -0.825 & 2.001 \end{bmatrix}$$
$$K_{12} = \begin{bmatrix} 0.825 & -2.001 \end{bmatrix}$$
$$K_{21} = \begin{bmatrix} -0.825 & 2.001 \end{bmatrix}$$
$$K_{22} = \begin{bmatrix} 0.825 & -2.001 \end{bmatrix}$$

In fact, this controller has $K_{11} = K_{21}$ and $K_{12} = K_{22}$; this agrees with the intuition that the controller should select a control effort based on the mode the system will take when that effort reaches the system.

Example 4.2. Consider the discrete inclusion specified by

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & .25 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} .25 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

While both modes of this system are individually controllable, the structure of B_1 and B_2 are such that the one-step reachable subspace is not all of $\mathbb{R}^2$. This prevents stabilization as the system can switch modes to prevent and control plan requiring more than one step to be completed successfully. With the addition of a one-step horizon, the system becomes controllable with gains

$$K_{11} = \begin{bmatrix} -1.28 & 0.077 \end{bmatrix}$$
$$K_{12} = \begin{bmatrix} -0.72 & 0.17 \end{bmatrix}$$
$$K_{21} = \begin{bmatrix} -0.17 & 0.72 \end{bmatrix}$$
$$K_{22} = \begin{bmatrix} -0.077 & 1.28 \end{bmatrix}$$

Chapter 5

Conclusions

In this thesis the uniform stabilization of switched linear systems was examined for a finite-path dependent controller with access to a finite future knowledge of the switching signal. Exact conditions were developed for both the state-feedback and output-feedback case in the form of increasing families of linear matrix inequalities. These conditions are a generalization of those developed previously in [14] for uniform stabilization when the controller is permitted only memory of past states. It was also shown that increasing the future look-ahead does not make the controller synthesis problem harder. Conversely, example systems were presented for which the finite horizon is required to stabilize the system.

An important difficulty in applying this result to controller synthesis is the effort needed to construct and solve large-scale linear matrix inequalities like the ones specified in this thesis. In order to effectively apply this and future related results to controller design, a comprehensive software toolbox should be developed which streamlines the specification of a system and the search for a stabilizing controller. The additional implementation of a distributed version of these tools will allow for very large-scale problems to be solved efficiently.

Unaddressed issues on this topic include the existence and synthesis of uniformly stabilizing controllers which also meet a specified level of disturbance attenuation. This will provide a generalization of the disturbance attenuation characterized in [13] and will proceed in fundamentally the same way. Such a result will be the topic of future work. As this work considers H_∞ disturbance attenuation, the related question of H_2 stabilization may also be considered.

The results on the stability of autonomous systems mentioned the possibility that sparse solutions to the required Lyapunov inequalities may exist. This amounts to constructing a switching graph which can generate exactly the admissible sequences but that does not assign one mode to each feasible finite switching path. The systematic search of graphs generated from the sets of paths of length $M + H$ are a certain class of such path-complete graphs but are not exhaustive in all cases. Further examination is required to determine if sets whose cardinality lies between those corresponding to fixed path lengths can be constructed and checked in a systematic way as well as whether such a search provides any benefit over the existing process.

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